Journal of Algebra 320 (2008) 3383–3397

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Nonassociative unitary Banach algebras ☆

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ARTICLE INFO

Article history:
Received 20 September 2006
Available online 4 September 2008
Communicated by Efim Zelmanov

Keywords:
Unitary normed algebra
Jordan algebra
Alternative algebra

ABSTRACT

We generalize the theory of associative unitary normed algebras to the setting of noncommutative Jordan algebras. Special attention is devoted to the case of alternative algebras.

1. Introduction

By a normed algebra we mean a real or complex (possibly nonassociative) algebra A endowed with a norm $\|\cdot\|$ satisfying $\|xy\| \leqslant \|x\|\|y\|$ for all $x, y \in A$. A complete normed associative algebra will be called a Banach algebra. A normed algebra is called norm-unital if it has a unit $\mathbf{1}$ such that $\|\mathbf{1}\| = 1$. Unitary elements of a norm-unital normed associative algebra A are defined as those invertible elements u of A satisfying $\|u\| = \|u^{-1}\| = 1$. By a unitary normed associative algebra we mean a norm-unital associative normed algebra A such that the convex hull of the set of its unitary elements is norm-dense in the closed unit ball of A. In the sequel we will denote by U_A the set of unitary elements of A. Relevant examples of unitary Banach algebras are all unital C^*-algebras and the discrete group algebras $\ell_1(G)$ for every group G.

The study of unitary Banach algebras is quite recent (see [1–3,5,10,11,13,31]). They were first considered by E.R. Cowie in her PhD thesis [10]. However, with the exception of some facts concerning discrete group algebras [11], her results were not published elsewhere. Fifteen years later, unitary Banach algebras were reconsidered by M.L. Hansen and R.V. Kadison [13], who were unaware of Cowie's

☆ Partially supported by Junta de Andalucía grants FQM 0199 and FQM 1215, and Project I+D MCYT MTM-2004-03882 and MTM-2006-15546-C02-02.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.07.021

work. Both [10] and [13] were mainly concerned with the achievement of characterizations of unital C^*-algebras among unitary Banach algebras. Recently, G.V. Wood [31] recovers some of Cowie's unpublished results, surveys the Hansen–Kadison paper, and proves some new results about discrete group algebras. In [2,5], complex Banach spaces whose algebras of operators are unitary are studied, and it is proved that, under certain additional conditions, they turn out to be Hilbert spaces. In [3], unitary Banach algebras are considered by themselves, showing that all unitary Banach algebras are quotients of discrete group algebras, proving different characterizations of them in terms of numerical ranges, studying dentability of their closed unit balls, and characterizing unital C^*-algebras among them by means of holomorphic conditions.

In [1], we continued the line of [3]. Indeed, we revisited the concepts of maximality and unique maximality, which are closely related to that of unitarity (see [10,11,13]), introduced the notions of strong maximality and strong unique maximality, and clarified how all these notions are related, as well as with that of unitarity. To this end, we also introduced the concept of minimality of the equivalent norm (a weakening of the classical notion of minimality of the norm [6]), and proved that a norm-unital associative normed algebra is uniquely maximal (respectively, strongly uniquely maximal) if and only if it is unitary and has minimality of the equivalent norm (respectively, minimality of the norm). Consequently, from the known facts that unital C^*-algebras are unitary (by the Russo–Dye theorem) and have minimality of the norm, we deduced that they are strongly uniquely maximal. Moreover, generalizing to the real case a result for complex algebras, first proved in [10] (see also [31]), we showed in [1] the following fact:

($\mathcal{F}$) Every maximal semisimple finite-dimensional real Banach algebra is isometrically isomorphic to a real C^*-algebra.

We also showed in [1] that commutative semisimple unitary Banach algebras satisfy Property ($\mathcal{S}$) which follows:

($\mathcal{S}$) There exists an algebra involution on the algebra, which is linear in the real case and conjugate-linear in the complex one, and maps each unitary element to its inverse.

Even, we could expect all semisimple unitary Banach algebras to satisfy Property ($\mathcal{S}$). Indeed, we proved that such a conjecture is equivalent to the one that every group G is "good" (which means that all primitive ideals of the complex Banach $*$-algebra $\ell_1(G)$ are $*$-invariant).

In the present paper, we are concerned with the generalization of the theory of unitary normed algebras to the nonassociative setting. Such a generalization is mainly motivated by the Russo–Dye-type theorem for unital JB^*-algebras, proved by J.D.M. Wright and M.A. Youngson [32]. Although non-commutative JB^*-algebras are "nearly" associative (they are in fact non-commutative Jordan algebras in the sense of [20]), in a very precise sense they become the largest nonassociative generalizations of (associative) C^*-algebras. Indeed, it is proved in [17] that an associative (respectively, nonassociative) normed complex algebra is a C^*-algebra (respectively, a non-commutative JB^*-algebra) if and only if it has an approximate unit bounded by one, and its open unit ball is a bounded symmetric domain (equivalently, the normed space of the algebra is linearly isometric to a JB^*-triple [19]). In view of the above comments, and due to the fact that the setting of unital non-commutative Jordan algebras becomes the largest nonassociative one where a notion of invertible element works reasonably [21], we restrict our attention to norm-unital normed non-commutative Jordan algebras. Unitary elements of such an algebra are defined verbatim as in the associative case, and the notions of unitarity, maximality, strong maximality, unique maximality, and strong unique maximality are translated literally from the associative setting to the more general one. Since the set of all unitary elements of a norm-unital normed non-commutative Jordan algebra need not be multiplicatively closed, we introduce weakly unitary normed non-commutative Jordan algebras as those norm-unital normed non-commutative Jordan algebras A such that the convex multiplicatively closed hull of U_A is dense in the closed unit ball of A. Replacing unitarity with weak unitarity, most results obtained in [1] remain true in the new setting (see Proposition 2.3 and Corollary 2.5). As a consequence, unital non-commutative JB^*-algebras turn out to be strongly uniquely maximal (Proposition 2.10).

On the other hand, weakly unitary norm-unital closed subalgebras of non-commutative JB^*-algebras are non-commutative JB^*-algebras (Theorem 2.11). The result just quoted becomes a nonassociative generalization of [13, Theorem 4]. Alternative algebras (respectively, alternative C^*-algebras) are very particular examples of non-commutative Jordan algebras (respectively, non-commutative JB^*-algebras). It is worth mentioning that, as in the particular associative case, for a norm-unital normed alternative algebra A, the set U_A is multiplicatively closed, and hence the concepts of unitarity and weak unitarity are equivalent for A. We prove that every finite-dimensional maximal unitary normed alternative complex algebra is isometrically isomorphic to an alternative C^*-algebra (Theorem 2.12). This generalizes [13, Theorem 6] (see also [3, Corollary 2.7]). Moreover, we prove a variant of Fact $(\mathcal{F})$ (reviewed some paragraphs ago) in the case of complex alternative algebras (see Theorem 2.16). Due to the lack of associativity, the proof of such a variant becomes the hardest one in the paper.

In the last section (Section 3) we introduce real non-commutative JB^*-algebras and real alternative C^*-algebras, and extend to the real case some results of the previous section. Among them, we emphasize the variant of Theorem 2.12 (reviewed in the preceding paragraph) for real algebras (see Theorem 3.10). It is also worth mentioning the fact that every group is good if and only if every unitary semisimple complete normed complex alternative algebra satisfies Property $(\mathcal{S})$, if and only if every unitary semisimple complete normed real alternative algebra satisfies Property $(\mathcal{S})$ (Proposition 3.11).

Notation. Given a vector space X, we denote by $L(X)$ the algebra of all linear operators on X. Now, let X be a real or complex normed space. Then the symbols $\mathcal{L}(X)$ and $\mathcal{G}_X$ will stand for the normed algebra of all bounded linear operators on X and the group of all surjective linear isometries on X, respectively. We note that the equality $\mathcal{G}_X = U_{\mathcal{L}(X)}$ holds. We denote by B_X and S_X the closed unit ball and the unit sphere, respectively, of X.

2. Nonassociative unitary Banach algebras

Given an algebra A, we denote by A^+ the algebra consisting of the vector space of A and the product $x \cdot y := \frac{xy+yx}{2}$. Following [29, p. 141], we define non-commutative Jordan algebras as those algebras satisfying the "Jordan identity" $(xy)x^2 = x(yx^2)$ and the "flexibility" condition $(xy)x = x(yx)$. Non-commutative Jordan algebras which are commutative are simply called Jordan algebras. An algebra A is a non-commutative Jordan algebra if and only if it is flexible and A^+ is a Jordan algebra (see again [29, p. 141]). Let A be a unital non-commutative Jordan algebra, and let x be an element of A. Following [21], we say that x is invertible in A if there exists y in A such that the equalities $xy = yx = \mathbf{1}$ and $x^2y = yx^2 = x$ hold. If x is invertible in A, then the element y above is unique, it is called the inverse of x, and is denoted by x^{-1}. Moreover x is invertible in A if and only if it is invertible in the Jordan algebra A^+. This reduces most questions and results on inverses in non-commutative Jordan algebras to the commutative case. For this particular case, the reader is referred to [16, Section I.11].

Let A be a normed non-commutative Jordan algebra. Then A is power-associative [29, p. 141] (i.e., all single-generated subalgebras are associative), and hence we can consider, as in the associative case, the spectral radius mapping $r_A(\cdot)$ defined by $r_A(x) := \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ for every $x \in A$. Assume that A is norm-unital. Then unitary elements of A are defined verbatim as in the associative case, and the symbol U_A will remain to denote the set of all unitary elements of A, and A is said to be unitary if the convex hull of U_A is dense in B_A. Similarly, the meanings of "maximal," "strongly maximal," "uniquely maximal," or "strongly uniquely maximal" for A are translated verbatim from the particular associative case to the new one. Indeed, we say that A is $\left\{\begin{array}{c}\text{maximal}\\ \text{strongly maximal}\\ \text{uniquely maximal}\\ \text{strongly uniquely maximal}\end{array}\right\}$ if, whenever

$|||\cdot|||$ is an $\left\{\begin{array}{c}\text{equivalent}\\ \text{continuous}\\ \text{equivalent}\\ \text{continuous}\end{array}\right\}$ norm on A converting A into a norm-unital normed algebra and satisfying

$U_A \subseteq U_{(A,|||\cdot|||)}$, we have that $\left\{ \begin{array}{c} U_A = U_{(A,|||\cdot|||)} \\ U_A = U_{(A,|||\cdot|||)} \\ \|\cdot\| = |||\cdot||| \\ \|\cdot\| = |||\cdot||| \end{array} \right\}$. The implications

$$\begin{tikzcd}[row sep=small] & \text{unique maximality} \arrow[dr, Rightarrow] & \\ \text{strong unique maximality} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{maximality} \\ & \text{strong maximality} \arrow[ur, Rightarrow] & \end{tikzcd} \tag{2.1}$$

are clear.

Proposition 2.1. *Let A be a norm-unital complete normed non-commutative Jordan complex algebra. Then the following assertions are equivalent*:

(1) *A is isometrically isomorphic to a commutative C^*-algebra.*
(2) *A is maximal and unitary, and there exists $k > 0$ such that $\|\cdot\| \leqslant k r_A(\cdot)$.*

Proof. According to [25, Proposition 31], the requirement $\|\cdot\| \leqslant k r_A(\cdot)$ for some $k > 0$ implies that A is associative. Now apply [1, Corollary 3.5]. $\square$

The development of a theory of unitary normed non-commutative Jordan algebras, similar to that in the associative case, stumbles on severe handicaps. Indeed, the set of all unitary elements of such an algebra need not be multiplicatively closed, and multiplications by unitary elements need not be isometries. These pathologies arise even in the nontrivial simplest cases, as is the one of the unitary complete normed Jordan algebra $A := B^+$, where B stands for the C^*-algebra of all 2×2 complex matrices. Here we have used the convention that, if B is a normed algebra, then B^+ is considered without notice as a new normed algebra under the norm of B. We have also kept in mind that, if B is a norm-unital normed non-commutative Jordan algebra, then we have clearly $U_B = U_{B^+}$, and hence B is unitary if and only if so is B^+.

Let A be a norm-unital normed non-commutative Jordan algebra. We denote by V_A the multiplicatively closed subset of A generated by U_A, so that, since B_A is multiplicatively closed, we have $V_A \subseteq B_A$. We say that A is weakly unitary if $\overline{co}(V_A) = B_A$.

Proposition 2.2. *Let A be a norm-unital normed non-commutative Jordan algebra. Then we have $co(V_{A^+}) \subseteq co(V_A)$. Therefore, if A^+ is weakly unitary, then A is weakly unitary.*

Proof. Since $co(V_A)$ is a convex multiplicatively closed subset of A, for $x, y \in co(V_A)$ we have $x \cdot y = \frac{1}{2}(xy + yx) \in co(V_A)$. Thus $co(V_A)$ is a multiplicatively closed subset of A^+ containing U_A. Since $U_A = U_{A^+}$, it follows that $co(V_A)$ contains V_{A^+}. $\square$

Proposition 2.3. *Let A be a norm-unital normed non-commutative Jordan algebra. Then the following conditions are equivalent*:

(1) *A is weakly unitary.*
(2) *For every continuous norm $|||\cdot|||$ on A satisfying*
 (a) *$(A, |||\cdot|||)$ is a norm-unital normed algebra, and*
 (b) *$U_A \subseteq U_{(A,|||\cdot|||)}$,*
 we have $|||\cdot||| \leqslant \|\cdot\|$.
(3) *For every equivalent norm $|||\cdot|||$ on A satisfying* (a) *and* (b) *above, we have $|||\cdot||| \leqslant \|\cdot\|$.*
(4) *For every continuous norm $|||\cdot|||$ on A satisfying* (a), (b) *above, and*
 (c) *$\|\cdot\| \leqslant |||\cdot|||$,*
 we have $|||\cdot||| = \|\cdot\|$.

Proof. (1) $\Rightarrow$ (2). Let $|||\cdot|||$ be a continuous norm on A satisfying (a) and (b). Then $B_{(A,|||\cdot|||)}$ is $\|\cdot\|$-closed and multiplicatively closed, and hence, by the assumption (1), we have

$$B_A = \overline{co}(V_A) \subseteq \overline{co}V_{(A,|||\cdot|||)} \subseteq B_{(A,|||\cdot|||)},$$

which implies $|||\cdot||| \leqslant \|\cdot\|$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). These implications are clear.

(4) $\Rightarrow$ (1). We follow with minor changes the proof of the implication (vii) $\Rightarrow$ (vi) in [3, Theorem 3.8]. Let $0 < \varepsilon \leqslant 1$. Since $co\,[(\varepsilon B_A) \cup V_A]$ is an absolutely convex subset of A contained in B_A and containing εB_A, the Minkowski functional of $co\,[(\varepsilon B_A) \cup V_A]$ (say $|||\cdot|||_\varepsilon$) is a norm on A satisfying

$$\varepsilon|||\cdot|||_\varepsilon \leqslant \|\cdot\| \leqslant |||\cdot|||_\varepsilon \tag{2.2}$$

and

$$\big\{a \in A\colon\ |||a|||_\varepsilon < 1\big\} \subseteq co\big[(\varepsilon B_A) \cup V_A\big] \subseteq \big\{a \in A\colon\ |||a|||_\varepsilon \leqslant 1\big\}. \tag{2.3}$$

On the other hand, since $(\varepsilon B_A) \cup V_A$ is multiplicatively closed, and the convex hull of a multiplicatively closed subset is multiplicatively closed, we deduce that $|||\cdot|||_\varepsilon$ actually becomes an algebra norm on A (argue as in [7, Proposition 1.9]). Now, if u is in U_A, then, by (2.2) and the right inclusion in (2.3), we have

$$1 = \|\mathbf{1}\| \leqslant |||\mathbf{1}|||_\varepsilon = \big|\big|\big|uu^{-1}\big|\big|\big|_\varepsilon \leqslant |||u|||_\varepsilon \big|\big|\big|u^{-1}\big|\big|\big|_\varepsilon \leqslant 1.1 = 1,$$

and hence $|||\mathbf{1}|||_\varepsilon = |||u|||_\varepsilon = |||u^{-1}|||_\varepsilon = 1$. Therefore the normed algebra $(A, |||\cdot|||_\varepsilon)$ is norm-unital, and the inclusion $U_A \subseteq U_{(A,|||\cdot|||_\varepsilon)}$ holds. Since $|||\cdot|||_\varepsilon$ is a continuous norm on A with $\|\cdot\| \leqslant |||\cdot|||_\varepsilon$ (by (2.2)), it follows from the assumption (4) that $|||\cdot|||_\varepsilon = \|\cdot\|$. Let x be in A with $\|x\| < 1$. It follows from the left inclusion in (2.3) that x belongs to $co\,[(\varepsilon B_A) \cup V_A]$. Since $co\,[(\varepsilon B_A) \cup V_A]$ is contained in $\varepsilon B_A + co\,(V_A)$, there exists y in $co\,(V_A)$ such that $\|x - y\| \leqslant \varepsilon$. The arbitrariness of $\varepsilon \in]0, 1]$ and $x \in \overset{\circ}{B}_A$, yields $\overset{\circ}{B}_A \subseteq \overline{co}\,(V_A)$. Therefore we have $\overline{co}\,V_A = B_A$, that is A is weakly unitary. $\square$

Remark 2.4. *Let A be a norm-unital normed non-commutative Jordan algebra.* It follows from the equivalence (1) $\Leftrightarrow$ (3) in Proposition 2.3 (respectively, from the definitions of maximality, strong maximality, unique maximality, or strong unique maximality) that, *if A^+ is weakly unitary* (*respectively, maximal, strongly maximal, uniquely maximal, or strongly uniquely maximal*), *then A is weakly unitary* (*respectively, maximal, strongly maximal, uniquely maximal, or strongly uniquely maximal*). Note that the part of the above assertion concerning weak unitarity has been previously proved in Proposition 2.2 without involving Proposition 2.3.

Let A be a normed algebra. We say that A has minimality of the $\left\{\begin{array}{c}\text{norm}\\ \text{equivalent norm}\end{array}\right\}$ if, for every $\left\{\begin{array}{c}\text{algebra}\\ \text{equivalent algebra}\end{array}\right\}$ norm $|||\cdot|||$ on A satisfying $|||\cdot||| \leqslant \|\cdot\|$, we have $|||\cdot||| = \|\cdot\|$. The implication

$$\text{minimality of the norm} \quad \Rightarrow \quad \text{minimality of the equivalent norm} \tag{2.4}$$

is clear. The following corollary follows from Proposition 2.2 in the same way as [1, Corollary 2.2] follows from [1, Proposition 2.1].

Corollary 2.5. *Let A be a norm-unital normed non-commutative Jordan algebra. Then we have*:

(1) *A is uniquely maximal if and only if it is weakly unitary and has minimality of the equivalent norm.*
(2) *A is strongly uniquely maximal if and only if it is weakly unitary and has minimality of the norm.*

Alternative algebras are defined as those algebras A satisfying the "left alternative law" $x^2y = x(xy)$ and the "right alternative law" $yx^2 = (yx)x$. We note for later reference that the left alternative law can be written as

$$L_{x^2} = L_x^2, \tag{2.5}$$

and hence, by linearization, as

$$L_{x \cdot y} = L_x \cdot L_y. \tag{2.6}$$

By Artin's theorem [29, p. 29], an algebra A is alternative (if and) only if, for all $x, y \in A$, the subalgebra of A generated by $\{x, y\}$ is associative. Artin's theorem implies that alternative algebras are non-commutative Jordan algebras, and that the inverse y of an invertible element x in a unital alternative algebra is characterized by the familiar condition $xy = yx = \mathbf{1}$. Moreover, if A is a unital alternative algebra, and if x, y are invertible elements of A, then xy is invertible with

$$(xy)^{-1} = y^{-1}x^{-1}, \tag{2.7}$$

and L_x (respectively, R_x) is a bijective operator on A with $L_x^{-1} = L_{x^{-1}}$ (respectively, $R_x^{-1} = R_{x^{-1}}$) [34, pp. 204–205]. These facts lead straightforwardly to the following.

Lemma 2.6. *Let A be a norm-unital normed alternative algebra. Then U_A is a multiplicative closed subset of A. Moreover, for every $u \in U_A$, the operators L_u and R_u are surjective isometries on A.*

It follows from the first conclusion in Lemma 2.6 that a normed alternative algebra is unitary if (and only if) it is weakly unitary. Therefore, keeping in mind the last conclusion in Proposition 2.2, we deduce the following.

Corollary 2.7. *Let A be a norm-unital normed alternative algebra. Then A is unitary* (*equivalently, A^+ is unitary*) *if and only if A^+ is weakly unitary.*

The following theorem is a variant of [1, Theorem 2.5] in the setting of alternative algebras. As its forerunner in [1], it improves [10, Corollary 8.15] (see also [31, Theorem 11]).

Theorem 2.8. *Let A be a norm-unital normed alternative algebra such that A^+ has minimality of the equivalent norm, and let M be a closed ideal of A. Then, for every $u \in M$ we have $\|u\| = \sup\{\|uv\|: v \in B_M\}$.*

Proof. Let $\pi : A \to A/M$ be the natural quotient homomorphism, and consider the equivalent vector space norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on A defined by $\|x\|_1 := \|x\| + \|\pi(x)\|$ and $\|x\|_2 := \|L_x\|_1$. It follows from (2.6), that $(A^+, \|\cdot\|_2)$ is a norm-unital normed algebra. Moreover, for $x, y \in A$, we have

$$\|xy\|_1 = \|xy\| + \big\|\pi(x)\pi(y)\big\| \leqslant \|x\|\|y\| + \big\|\pi(x)\big\|\big\|\pi(y)\big\| \leqslant \|x\|\|y\|_1,$$

and hence $\|x\|_2 \leqslant \|x\|$. Since A^+ has minimality of the equivalent norm, we deduce that $\|\cdot\|_2 = \|\cdot\|$. Now the proof is concluded by repeating verbatim the corresponding part of the argument in the associative case (see the proof of [1, Theorem 2.5]). $\square$

By a non-commutative JB^*-algebra we mean a complete normed non-commutative Jordan complex algebra (say A) endowed with a conjugate-linear algebra involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every x in A. Here, for $x \in A$, U_x stands for the mapping $y \to x(xy + yx) - x^2 y$ from A to A. Non-commutative JB^*-algebras which are commutative are simply called JB^*-algebras. If A is a non-commutative JB^*-algebra, then it follows from the equality $U_x(y) = 2x \cdot (x \cdot y) - x^2 \cdot y$ (which is true for all $x, y \in A$) that A^+ becomes naturally a JB^*-algebra. This fact allows us to reduce many questions and results concerning non-commutative JB^*-algebras to the commutative case.

Lemma 2.9. *(See [8, Proposition 4.3].) Let A be a unital non-commutative JB^*-algebra. Then unitary elements of A are precisely those invertible elements u in A satisfying $u^{-1} = u^*$.*

Proposition 2.10. *Let A be a unital non-commutative JB^*-algebra. Then A is unitary and strongly uniquely maximal.*

Proof. That A is unitary follows from Lemma 2.9 and [32]. Then, that A is strongly uniquely maximal follows from Corollary 2.5 and the fact that A has minimality of the norm [24, Proposition 11]. □

Theorem 2.11. *Every weakly unitary norm-unital closed subalgebra of a non-commutative JB^*-algebra is a non-commutative JB^*-algebra.*

Proof. Let A be a non-commutative JB^*-algebra, and let B be a weakly unitary norm-unital closed subalgebra of A. It is enough to show that B is $*$-invariant. To this end, note that the unit (say $\mathbf{1}$) of B is a norm-one idempotent of A, and hence, by [17, Lemma 2.2], we have $\mathbf{1}^* = \mathbf{1}$. Therefore, keeping in mind [20, p. 188], the set $C := \{x \in A\colon\ x\mathbf{1} = \mathbf{1}x = x\}$ becomes a closed $*$-invariant subalgebra of A which contains B, and whose unit is $\mathbf{1}$. Thus, replacing A with C if necessary, we may assume that $\mathbf{1}$ is in fact a unit for A. Then we have $U_B \subseteq U_A$, so U_B is a $*$-invariant subset of A (by Lemma 2.9), and so V_B is also $*$-invariant. Since $*$ is continuous, and $B_B = \overline{co}(V_B)$, we deduce that B is $*$-invariant, as required. □

Note that, in the above proof, the assumption that B is weakly unitary can be relaxed to the one that B is equal to the closed linear hull of V_B.

By an alternative C^*-algebra we mean a complete normed alternative complex algebra (say A) with a conjugate-linear algebra-involution $*$ satisfying $\|x^*x\| = \|x\|^2$ for every x in A. Since, for elements x, y in an alternative algebra, the equality $U_x(y) = xyx$ holds, it is not difficult to realize that alternative C^*-algebras become particular examples of non-commutative JB^*-algebras. In fact alternative C^*-algebras are precisely those non-commutative JB^*-algebras which are alternative [22, Proposition 1.3]. The following theorem generalizes and refines [13, Theorem 6].

Theorem 2.12. *Let A be a norm-unital normed finite-dimensional alternative complex algebra such that A is equal to the linear hull of U_A. Then there exists a conjugate-linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$. Moreover, endowed with such an involution, A is $*$-isomorphic to an alternative C^*-algebra. If in addition A is maximal, then A is in fact an alternative C^*-algebra.*

Proof. By Lemma 2.6 and [28, Theorem 9.5.1], there exists an inner product $(\cdot|\cdot)$ on A such that L_u belongs to $\mathcal{G}_{(A,(\cdot|\cdot))}$ whenever u is in U_A. Then, for $u \in U_A$, we have $L_u^* = L_u^{-1} = L_{u^{-1}}$, where, for $T \in L(A)$, T^* denotes the adjoint of T relative to $(\cdot|\cdot)$. Therefore, if $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $u_1, \ldots, u_n \in U_A$ are such that $\sum_{k=1}^n \lambda_k u_k = 0$, then we have

$$L_{\sum_{k=1}^n \overline{\lambda_k} u_k^{-1}} = \sum_{k=1}^n \overline{\lambda_k} L_{u_k^{-1}} = \sum_{k=1}^n \overline{\lambda_k} L_{u_k}^* = \left(\sum_{k=1}^n \lambda_k L_{u_k}\right)^* = (L_{\sum_{k=1}^n \lambda_k u_k})^* = 0,$$

and hence $\sum_{k=1}^{n} \overline{\lambda_k} u_k^{-1} = 0$. It follows that

$$x = \sum_{k=1}^{n} \lambda_k u_k \to x^* := \sum_{k=1}^{n} \overline{\lambda_k} u_k^{-1}$$

(with $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $u_1, \ldots, u_n \in U_A$) is a well-defined mapping from A to A, which, in view of (2.7), becomes a conjugate-linear algebra involution on A satisfying

$$L_x^* = L_{x^*} \tag{2.8}$$

for every $x \in A$, and

$$u^* = u^{-1} \tag{2.9}$$

whenever u lies in U_A. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the vector space norms on A defined by $\|x\|_1 := \sqrt{(x|x)}$ and $\|x\|_2 := \|L_x\|_1$. It follows from (2.6) and (2.8) that the mapping $x \to L_x$ is an isometric $*$-homomorphism from $(A^+, *, \|\cdot\|_2)$ to $\mathcal{L}((A, \|\cdot\|_1))^+$. Therefore, since $\mathcal{L}((A, \|\cdot\|_1))$ is a C^*-algebra, $(A^+, *, \|\cdot\|_2)$ is a JB^*-algebra. Since $*$ is an algebra involution on A, it follows from [26, Theorem 1] that $(A, *, \|\cdot\|_2)$ is an alternative C^*-algebra.

Assume that A is maximal. Then, since $U_A \subseteq U_{(A,\|\cdot\|_2)}$ (by (2.9) and Lemma 2.9), we have $U_A = U_{(A,\|\cdot\|_2)}$. Since $(A, \|\cdot\|_2)$ is uniquely maximal (by Proposition 2.10), we have in fact $\|\cdot\| = \|\cdot\|_2$. □

The following lemma is a byproduct of the proof of [9, Theorem 2.11].

Lemma 2.13. *Let X be a finite-dimensional complex vector space, and let g be a nondegenerate symmetric bilinear form on X. Then, given an arbitrary inner product $\langle\cdot|\cdot\rangle$ on X, we have:*

(1) *There exists a unique bijective conjugate-linear mapping $\sigma : X \to X$ satisfying $g(x, y) = \langle x|\sigma(y)\rangle$ for all $x, y \in X$.*
(2) *The bijective linear operator $F := \sigma^2$ on $(X, \langle\cdot|\cdot\rangle)$ is positive, and hence the mapping $(\cdot|\cdot) : X \times X \to \mathbb{C}$, defined by*

$$(x|y) := \left\langle F^{\frac{1}{2}}(x)|y\right\rangle,$$

is an inner product on X.
(3) *The mapping $* := F^{-\frac{1}{2}} \circ \sigma$ is an isometric conjugate-linear involution on $(X, (\cdot|\cdot))$ satisfying $g(x, y) = (x|y^*)$ for all $x, y \in X$.*

Let X be a finite-dimensional vector space, and let g be a nondegenerate symmetric bilinear form g on X. For T in the algebra $L(X)$ of all linear operators on X, we denote by $T^\sharp$ the unique element in $L(X)$ satisfying $g(T(x), y) = g(x, T^\sharp(y))$ for all $x, y \in X$, and we recall that the mapping $T \to T^\sharp$ is a linear algebra involution on $L(X)$.

Corollary 2.14. *Let X be a finite-dimensional complex Banach space, and let g be a nondegenerate symmetric bilinear form on X. Then there exists an inner product $(\cdot|\cdot)$ on X, and an isometric conjugate-linear involution $*$ on $(X, (\cdot|\cdot))$ satisfying:*

(1) *$g(x, y) = (x|y^*)$ for all $x, y \in X$.*
(2) *$\mathcal{G}_X \cap \mathcal{G}_X^\sharp \subseteq \mathcal{G}_{(X,(\cdot|\cdot))}$, where $\mathcal{G}_X^\sharp := \{T^\sharp\colon\ T \in \mathcal{G}_X\}$.*

Proof. By [28, Theorem 9.5.1], there exists an inner product $\langle\cdot|\cdot\rangle$ on X such that $\mathcal{G}_X \subseteq \mathcal{G}_{(X,\langle\cdot|\cdot\rangle)}$. Let σ, F, $(\cdot|\cdot)$, and $*$ be the mappings corresponding to $\langle\cdot|\cdot\rangle$ via Lemma 2.13. Then, by that lemma, $(\cdot|\cdot)$ is an inner product on X, and $*$ is an isometric conjugate-linear involution on $(X,(\cdot|\cdot))$ satisfying condition (1) in the present corollary. Let T be in $\mathcal{G}_X \cap \mathcal{G}_X^\sharp$. Then, since T belongs to $\mathcal{G}_{(X,\langle\cdot|\cdot\rangle)}$, we have

$$\big\langle x|T^{-1}\big(\sigma(y)\big)\big\rangle = \big\langle T(x)|\sigma(y)\big\rangle = g\big(T(x), y\big) = g\big(x, T^\sharp(y)\big) = \big\langle x|\sigma\big(T^\sharp(y)\big)\big\rangle$$

for all $x, y \in X$, and hence $T^{-1}\circ\sigma = \sigma\circ T^\sharp$. Since $\mathcal{G}_X \cap \mathcal{G}_X^\sharp$ is a $\sharp$-invariant group of bijective operators on X, it follows

$$T\circ F = T\circ\sigma^2 = \sigma\circ\big(T^{-1}\big)^\sharp\circ\sigma = \sigma\circ\big(T^\sharp\big)^{-1}\circ\sigma = \sigma^2\circ\big(T^\sharp\big)^\sharp = F\circ T,$$

and hence $T\circ F^{\frac{1}{2}} = F^{\frac{1}{2}}\circ T$ (because $F^{\frac{1}{2}}$ is a limit of polynomials in F). Finally, applying again that T belongs to $\mathcal{G}_{(X,\langle\cdot|\cdot\rangle)}$, we have

$$\big(T(x)|T(x)\big) = \big\langle F^{\frac{1}{2}}\big(T(x)\big)|T(x)\big\rangle = \big\langle T\big(F^{\frac{1}{2}}(x)\big)|T(x)\big\rangle = \big\langle F^{\frac{1}{2}}(x)|x\big\rangle = (x|x)$$

for every $x \in X$, and hence T belongs to $\mathcal{G}_{(X,(\cdot|\cdot))}$. $\square$

We recall that a complex JB^*-triple is a complex Banach space X with a continuous triple product $\{\cdot\cdot\cdot\}: X\times X\times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(1) For all x in X, the mapping $y \to \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.
(2) The main identity

$$\big\{ab\{xyz\}\big\} = \big\{\{abx\}yz\big\} - \big\{x\{bay\}z\big\} + \big\{xy\{abz\}\big\}$$

holds for all a, b, x, y, z in X.
(3) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Concerning condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be hermitian if $\|\exp(irT)\| = 1$ for every r in $\mathbb{R}$. Up to isomorphisms, there exists a unique simple finite-dimensional alternative nonassociative complex algebra, which will be denoted by $\mathcal{C}$. We refer the reader to [29] for the fact just quoted, as well as for the remaining properties of $\mathcal{C}$ needed in our argument.

Proposition 2.15. *Let $\|\cdot\|$ be a vector space norm on $\mathcal{C}$. Then there exists a vector space norm $|||\cdot|||$ on $\mathcal{C}$, and a vector space involution $*$ on $\mathcal{C}$, satisfying*:

(1) *$(\mathcal{C}^+, |||\cdot|||, *)$ is a JB^*-algebra.*
(2) *$u^* = u^{-1}$ whenever u is in $\mathcal{C}$ such that L_u and R_u are isometries on $(\mathcal{C}, \|\cdot\|)$.*

Proof. There exists a unique linear algebra involution τ on $\mathcal{C}$ such that $x + \tau(x) \in \mathbb{C}\mathbf{1}$ and $x\tau(x) = \tau(x)x \in \mathbb{C}\mathbf{1}$ for every $x \in \mathcal{C}$. Therefore, if for $x \in \mathcal{C}$ we put $x + \tau(x) = 2t(x)\mathbf{1}$ and $x\tau(x) = n(x)\mathbf{1}$, with $t(x)$ and $n(x)$ in $\mathbb{C}$, then the mappings $t(\cdot)$ and $n(\cdot)$ are linear and quadratic, respectively, and we have

$$x^2 - 2t(x)x + n(x)\mathbf{1} = 0. \tag{2.10}$$

On the other hand, an element $x \in \mathcal{C}$ is invertible if and only if $n(x) \neq 0$, and, if this is the case, then

$$x^{-1} = n(x)^{-1}\tau(x). \tag{2.11}$$

Moreover, the mapping $g : (x, y) \to t(xy)$ becomes a nondegenerate symmetric bilinear form on $\mathcal{C}$ satisfying

$$g(xy, z) = g(x, yz) \tag{2.12}$$

and

$$g\big(\tau(x), y\big) = g\big(x, \tau(y)\big) \tag{2.13}$$

for all $x, y, z \in \mathcal{C}$.

Define a vector space norm $\|\cdot\|_1$ on $\mathcal{C}$ by

$$\|x\|_1 := \|x\| + \big\|\tau(x)\big\|. \tag{2.14}$$

Then, applying Corollary 2.14, we find an inner product $(\cdot|\cdot)$ on $\mathcal{C}$, and an isometric conjugate-linear vector space involution $*$ on $(\mathcal{C}, (\cdot|\cdot))$ satisfying:

$$g(x, y) = (x|y^*) \tag{2.15}$$

for all $x, y \in \mathcal{C}$, and

$$\mathcal{G}_{(\mathcal{C},\|\cdot\|_1)} \cap \mathcal{G}^{\sharp}_{(\mathcal{C},\|\cdot\|_1)} \subseteq \mathcal{G}_{(\mathcal{C},(\cdot|\cdot))}. \tag{2.16}$$

We note that, by (2.15), for T in $L(\mathcal{C})$, we have

$$T^{\bullet} = * \circ T^{\sharp} \circ *, \tag{2.17}$$

where $T^{\bullet}$ denotes the adjoint of T relative to $(\cdot|\cdot)$. Now, applying (2.13), (2.14), (2.16), and (2.17), we obtain that τ commutes with $*$, and hence that $\mathbb{C}\mathbf{1}$ (equal to the range of $1 + \tau$) is $*$-invariant. Therefore, since $*$ is isometric on $(\mathcal{C}, (\cdot|\cdot))$, we have $\mathbf{1}^* = \gamma\mathbf{1}$ for some $\gamma \in S_{\mathbb{C}}$. But, since

$$1 = t(\mathbf{1}) = g(\mathbf{1}, \mathbf{1}) = (\mathbf{1}|\mathbf{1}^*) = \overline{\gamma}(\mathbf{1}|\mathbf{1})$$

(by (2.15)), we have in fact $\gamma = 1$, and hence

$$\mathbf{1}^* = \mathbf{1}. \tag{2.18}$$

Put $U := \{u \in \mathcal{C}\colon\ \{L_u, R_u\} \subseteq \mathcal{G}_{(\mathcal{C},\|\cdot\|)}\}$. We claim that U is τ-invariant. To prove the claim, let us take u in U. Then L_u is a surjective linear isometry on a suitable complex Banach space, and satisfies $L_u^2 - 2t(u)L_u + n(u) = 0$ (by (2.10) and (2.5)). This implies that $|n(u)| = 1$ (because $n(u)$ is the product of the elements in the spectrum of L_u). Therefore, since $\tau(u) = n(u)u^{-1}$ (by (2.11)), and U is closed by passing to inverses and by multiplication of its elements by unimodular numbers, it follows that $\tau(u)$ lies in U. Now that the claim has been proved, it follows from (2.14) that L_u and R_u lie in $\mathcal{G}_{(\mathcal{C},\|\cdot\|_1)}$ whenever u belongs to U. Then, applying (2.12), (2.16), and (2.17), we obtain that $L_u^{-1} = * \circ R_u \circ *$ whenever u belongs to U, and hence, by (2.18), that $u^{-1} = L_{u^{-1}}(\mathbf{1}) = L_u^{-1}(\mathbf{1}) = (* \circ R_u \circ *)(\mathbf{1}) = u^*$. This proves condition (2) in the statement.

Let x and y be in $\mathcal{C}$. Then, since $x\tau(x) = n(x)\mathbf{1}$, we have $t(x\tau(x)) = n(x)$. This allows us to linearize (2.10) to obtain

$$x \cdot y - t(x)y - t(y)x + \frac{t(x\tau(y)) + t(y\tau(x))}{2}\mathbf{1} = 0.$$

Keeping in mind the definition of g, and invoking (2.13), (2.15), and (2.18), the equality above reads as

$$x \cdot y = (x|\mathbf{1})y + (y|\mathbf{1})x - \big(x|\tau(y)^*\big)\mathbf{1}. \tag{2.19}$$

Replacing in (2.19) x and y with y^* and x^*, respectively, keeping in mind that $*$ is an isometric conjugate-linear involution on $(\mathcal{C}, (\cdot|\cdot))$, and applying (2.18), we realize that $(x \cdot y)^* = y^* \cdot x^*$. Thus, $*$ is an algebra involution on $\mathcal{C}^+$. Put $\sigma := * \circ \tau$. Since τ is an isometry on $(\mathcal{C}, (\cdot|\cdot))$ (by (2.13), (2.14), and (2.16)), and commutes with $*$, σ becomes an isometric conjugate-linear vector space involution on $(\mathcal{C}, (\cdot|\cdot))$. Therefore $\mathcal{C}$ becomes a JB^*-triple under the triple product

$$\{xzy\} := (x|z)y + (y|z)x - \big(x|\sigma(y)\big)\sigma(z),$$

and a suitable norm $|||\cdot|||$ satisfying $|||\{xyz\}||| \leqslant |||x|||\,|||y|||\,|||z|||$ for all $x, y, z \in \mathcal{C}$, and $|||x|||^2 = (x|x)$ whenever x is in $\mathcal{C}$ with $\sigma(x) = x$ [30, Example 20.36]. Since $x \cdot y = \{x\mathbf{1}y\}$ (by (2.19)), and $(\mathbf{1}|\mathbf{1}) = 1$, it follows that $|||x \cdot y||| \leqslant |||x|||\,|||y|||$ for all $x, y, \in \mathcal{C}$. Thus, $|||\cdot|||$ is an algebra norm on $\mathcal{C}^+$. On the other hand, a straightforward computation, involving (2.19), shows that $U_x(x^*) = \{xxx\}$, so that we have $|||U_x(x^*)||| = |||x|||^3$ for every $x \in \mathcal{C}$. In this way, the proof of condition (1) in the statement is concluded. $\square$

Theorem 2.16. *Let A be a semisimple finite-dimensional norm-unital normed complex alternative algebra such that A^+ is maximal. Then A is (isometrically isomorphic to) an alternative C^*-algebra.*

Proof. By [29, Theorem 3.12], we have $A = \bigoplus_{i=1}^n A_i$, where, for $i = 1, \dots, n$, either $A_i = L(X_i)$ for some complex vector space X_i, or $A_i = \mathcal{C}$. In the first case, we know that there exists an involution $*_i$ and a norm $|||\cdot|||_i$ on A_i such that $(A_i, |||\cdot|||_i, *_i)$ becomes a C^*-algebra in such a way that $\pi_i(U_A) \subseteq U_{(A_i, |||\cdot|||_i)}$, where π_i stands for the projection from A onto A_i corresponding to the decomposition $A = \bigoplus_{i=1}^n A_i$ (see the proof of [1, Theorem 5.8]). In any case, for $u \in U_A$ and $x_i \in A_i$, we have $\pi_i(u)x_i = ux_i$ and $x_i\pi_i(u) = x_iu$, and hence, by Lemma 2.6, $\|\pi_i(u)x_i\| = \|x_i\|$ and $\|x_i\pi_i(u)\| = \|x_i\|$. It follows from Proposition 2.15 that, in the second case, there exists an involution $*_i$ and a norm $|||\cdot|||_i$ on A_i such that $(A_i^+, |||\cdot|||_i, *_i)$ becomes a JB^*-algebra in such a way that $\pi_i(U_A) \subseteq U_{(A_i^+, |||\cdot|||_i)}$. For $a = \sum_{i=1}^n a_i \in A$ with $a_i \in A_i$ for all i, put $|||a||| := \max\{|||a_i|||_i\colon\ i = 1, \dots, n\}$, and $a^* := \sum_{i=1}^n a_i^{*_i}$. It follows that $(A^+, |||\cdot|||, *)$ is a JB^*-algebra, and that $U_{A^+} = U_A \subseteq U_{(A^+, |||\cdot|||)}$. Since A^+ is maximal, we have in fact $U_{A^+} = U_{(A^+, |||\cdot|||)}$. Since $(A^+, |||\cdot|||)$ is uniquely maximal (by Proposition 2.10), we deduce $\|\cdot\| = |||\cdot|||$ on A. Now $\|\cdot\|$ is an algebra norm on A converting A^+ into a JB^*-algebra, so that, by [27, Corollary 1.2], A is an alternative C^*-algebra. $\square$

3. Real non-commutative JB^*-algebras

By a real non-commutative JB^*-algebra we mean a closed $*$-invariant real subalgebra of a (complex) non-commutative JB^*-algebra. If B is a non-commutative JB^*-algebra, and if τ is an involutive conjugate-linear $*$-automorphism of B, then the set $A := \{x \in B\colon\ \tau(x) = x\}$ is a closed $*$-invariant real subalgebra of B, and hence a real non-commutative JB^*-algebra. Note that, in this case, we have $B = A \oplus iA$, and therefore B is algebraically isomorphic to the complexification $\mathbb{C} \otimes A$ of A.

Lemma 3.1. *Let A be a real non-commutative JB^*-algebra. Then there exists a non-commutative JB^*-algebra B, and an involutive conjugate-linear $*$-automorphism τ of B, such that $A = \{x \in B\colon\ \tau(x) = x\}$.*

Proof. Let C be a non-commutative JB^*-algebra containing A as a closed $*$-invariant real subalgebra. Let $\bar{C}$ stand for a set-copy of C with operations and norm defined by $\bar{x}+\bar{y} := \overline{x+y}$, $\bar{x}\,\bar{y} := \overline{xy}$, $\lambda\bar{x} := \overline{\bar{\lambda}x}$ (where, for $\lambda \in \mathbb{C}$, $\bar{\lambda}$ means the conjugate of λ), $\bar{x}^* := \overline{x^*}$, and $\|\bar{x}\| := \|x\|$. Then $\bar{C}$ is a non-commutative JB^*-algebra, and hence $D := C \oplus_\infty \bar{C}$ is a non-commutative JB^*-algebra. Moreover, the mapping $\tau : (x, \bar{y}) \to (y, \bar{x})$ becomes an involutive conjugate-linear $*$-automorphism of D, and A can be identified with the closed $*$-invariant real subalgebra of D given by $\{(x, \bar{x}):\ x \in A\}$. Now, $B := A + iA$ is a closed $*$- and τ-invariant subalgebra of D, and we have $A = \{x \in B:\ \tau(x) = x\}$. $\square$

For a non-commutative Jordan algebra A, let $(x, y) \to U_{x,y}$ be the unique symmetric bilinear mapping from $A \times A$ to $L(A)$ satisfying $U_{x,x} = U_x$ for every $x \in A$. It is well known that, if A is a non-commutative JB^*-algebra, then A becomes a JB^*-triple under its own norm and the triple product $\{\cdots\}$ defined by $\{xyz\} := U_{x,z}(y^*)$ (see [8] and [33]). Now recall that, according to [15], real JB^*-triples are defined as closed real subtriples of (complex) JB^*-triples. It follows that real non-commutative JB^*-algebras are real JB^*-triples in a natural way. All facts just reviewed will be applied without notice in what follows (mainly, in the proof of Proposition 3.3 below). The following lemma follows from Lemma 3.1 and [8, Proposition 4.3 and Lemma 4.1].

Lemma 3.2. *Let A be a unital real non-commutative JB^*-algebra. Then U_A is closed in A, and every element of U_A is an extreme point of B_A.*

It is well known that C^*-algebras whose Banach space is reflexive are finite-dimensional. This fact is no longer true when non-commutative JB^*-algebras replace C^*-algebras. Anyway, non-commutative JB^*-algebras whose Banach space is reflexive have a unit, and are in fact Hilbertizable (i.e., their Banach spaces are isomorphic to Hilbert spaces) [23, Theorem 3.5]. It follows from Lemma 3.1 that real non-commutative JB^*-algebras whose Banach space is reflexive have a unit, and are Hilbertizable.

Proposition 3.3. *Let A be a Hilbertizable real non-commutative JB^*-algebra. Then the extreme points of B_A are precisely the unitary elements of A.*

Proof. By Lemma 3.2, unitary elements of A are extreme points of B_A. Let u be an extreme point of B_A. Consider the non-commutative JB^*-algebra B, and the involutive conjugate-linear $*$-automorphism τ of B, given by Lemma 3.1. By the proof of [15, Lemma 3.3], u is a "complex extreme point" of B_B, so u is an extreme point of B_B (by [8, Lemma 4.1]), and so, since B is Hilbertizable, u is a denting point of B_B (by [4, Theorem 4.1]). Since $B_B = \overline{co}(U_B)$ (by Proposition 2.10), it follows from [3, Lemma 4.2] and Lemma 3.2 that u belongs to U_B. Therefore u lies in $U_B \cap A = U_A$. $\square$

Corollary 3.4. *Let A be a Hilbertizable real non-commutative JB^*-algebra. Then A is unitary.*

Proof. By the Krein–Milman theorem and Proposition 3.3, the convex hull of U_A is weak-dense in B_A. But weak-closed convex subsets of A are norm-closed. $\square$

It is known that the topology of any algebra norm on a JB^*-algebra is stronger than that of the JB^*-norm [24, Theorem 10]. Keeping in mind this result and Lemma 3.1, we can argue as in the proof of [1, Lemma 5.2] to obtain the following.

Lemma 3.5. *Let A be a real non-commutative JB^*-algebra, and let $\|\cdot\|_1$ be an arbitrary algebra norm on A. Then the topology of $\|\cdot\|_1$ is stronger than that of the natural norm $\|\cdot\|$.*

By a real alternative C^*-algebra we mean a closed $*$-invariant real subalgebra of a (complex) alternative C^*-algebra. Since real alternative C^*-algebras are real non-commutative JB^*-algebras, we can argue as in the proof of [1, Corollary 5.3] (applying Lemma 3.5 instead of [1, Lemma 5.2]) to obtain the following.

Corollary 3.6. *Let A be a real alternative C^*-algebra. Then A has minimality of the norm.*

Now, putting together Corollaries 2.5, 3.4, and 3.6, we derive the following.

Corollary 3.7. *Let A be a finite-dimensional real alternative C^*-algebra. Then A is uniquely maximal.*

In Corollary 3.7 just formulated, we could have relaxed the requirement that A is finite-dimensional to the one that A is Hilbertizable. However, such a relaxing is only apparent. Indeed, it follows from Lemma 3.1 and [18, Remark 7.3] that real alternative C^*-algebras whose Banach space is reflexive are finite-dimensional.

The tensor product of two non-commutative Jordan algebras need not be a non-commutative Jordan algebra. Indeed, if $M_2(\mathbb{F}) \otimes A$ is flexible, for an algebra A over a field $\mathbb{F}$, then A is associative. Anyway, the tensor product $B \otimes A$ is an alternative (respectively, non-commutative Jordan) algebra whenever B is an associative commutative algebra, an A is an alternative (respectively, non-commutative Jordan) algebra. Moreover, as in the associative case, we have the following.

Proposition 3.8. *Let B be a norm-unital normed associative commutative algebra, and let A be a norm-unital normed non-commutative Jordan algebra, both over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If A and B are unitary, then the projective tensor product $B \otimes_\pi A$ is a unitary normed non-commutative Jordan algebra. If A is the closed linear hull of U_A, and if B is the closed linear hull of U_B, then $B \otimes_\pi A$ is the closed linear hull of $U_{B\otimes_\pi A}$.*

In particular, we have the following.

Corollary 3.9. *Let A be a norm-unital normed non-commutative Jordan real algebra. If A is unitary, then the normed complexification $\mathbb{C} \otimes_\pi A$ of A is unitary. If A is the closed linear hull of U_A, then $\mathbb{C} \otimes_\pi A$ is the closed linear hull of $U_{\mathbb{C}\otimes_\pi A}$.*

Now we are ready to prove the main result in this section.

Theorem 3.10. *Let A be a norm-unital normed finite-dimensional alternative real algebra such that A is equal to the linear hull of U_A. Then there exists a linear algebra involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$. Moreover, endowed with such an involution, A is $*$-isomorphic to a real alternative C^*-algebra. If in addition A is maximal, then A is in fact a real alternative C^*-algebra.*

Proof. By Corollary 3.9 and Theorem 2.12, there exists a norm $|||\cdot|||$ and an involution $*$ on $\mathbb{C} \otimes A$, such that $(\mathbb{C} \otimes A, |||\cdot|||, *)$ is an alternative C^*-algebra, and $u^* = u^{-1}$ for every $u \in U_{\mathbb{C}\otimes_\pi A}$. This last property of $*$ implies that A is $*$-invariant, and hence that $(A, |||\cdot|||, *)$ is a real alternative C^*-algebra.

Assume that A is maximal. Then, since $U_A \subseteq U_{(A,|||\cdot|||)}$, we have $U_A = U_{(A,|||\cdot|||)}$. Since $(A, |||\cdot|||)$ is uniquely maximal (by Corollary 3.7), we have in fact $\|\cdot\| = |||\cdot|||$ on A. □

Let A be a nonassociative algebra. Then maximal modular left ideals of A are defined verbatim as in the associative case, and primitive ideals of A are defined as the cores of maximal modular left ideals of A. Here, by the core of a given subspace X of A we mean the largest ideal of A contained in X. According to [7, Definition 24.11], the notion of primitive ideal just introduced agrees with the familiar one when A is associative. The radical of A is defined as the intersection of all primitive ideals of A, and A is said to be primitive (respectively, semisimple) if zero is a primitive ideal of A (respectively, if the radical of A is equal to zero). If A is complete normed, then, as in the associative case, maximal modular left ideals of A are closed, and hence primitive ideals of A are closed either.

In the case of non-commutative Jordan algebras, the notions of radical, primitivity, and semisimplicity, introduced above, are not subtle enough to allow the development of a satisfactory structure theory, and therefore they have been suitably refined in the literature (see [14,21], and [12]). Nevertheless, in the particular case of alternative algebras, such refinements are unnecessary [34, Theorem 10.4.5]. Moreover, primitive alternative algebras are either associative or unital simple eight-dimensional over their centers [34, Theorem 10.1.1]. It follows from the Gelfand–Mazur theorem that primitive alternative normed algebras are either associative or finite-dimensional.

Proposition 3.11 immediately below complement [1, Proposition 4.8]. Among other facts, its proof involves the one that, as in the associative case [3, Proposition 2.1], quotients of unitary normed non-commutative Jordan algebras are unitary.

Proposition 3.11. *The following assertions are equivalent*:

(1) *Every group is a good group.*
(2) *Every unitary semisimple complete normed real alternative algebra has an isometric linear algebra involution sending unitary elements to their inverses.*
(3) *The same as* (2)*, with primitive instead of semisimple*
(4) *Every unitary semisimple complete normed complex alternative algebra has an isometric conjugate-linear algebra involution sending unitary elements to their inverses.*
(5) *The same as* (4)*, with primitive instead of semisimple.*

Proof. (1) $\Rightarrow$ (3) (respectively, (1) $\Rightarrow$ (5)). Since primitive alternative normed algebras are associative or finite-dimensional, this implication follows from Theorem 3.10 (respectively, Theorem 2.12) and [1, Proposition 4.8]. When Theorems 2.12 and 3.10 are applied, note that, as in the associative case [3, Remark 2.9.(c)], continuous involutions on unitary normed non-commutative Jordan algebras, sending unitaries to their inverses, are isometries.

(3) $\Rightarrow$ (2) (respectively, (5) $\Rightarrow$ (4)). Let A be a unitary semisimple complete normed real (respectively, complex) alternative algebra. If $\{\lambda_u\}_{u\in U_A}$ is a family of real (respectively, complex) numbers satisfying $\sum_{u\in U_A}|\lambda_u|<\infty$ and $\sum_{u\in U_A}\lambda_u u=0$, then, for each primitive ideal P of A, we can consider the isometric linear (respectively, conjugate-linear) algebra involution $*$ on A/P sending unitary elements to their inverses, whose existence is assured by the assumption (3) (respectively, (5)), to have

$$\sum_{u\in U_A}\overline{\lambda_u}u^{-1}+P=\sum_{u\in U_A}\overline{\lambda_u}(u+P)^{-1}=\sum_{u\in U_A}\overline{\lambda_u}(u+P)^{*}=\left(\sum_{u\in U_A}\lambda_u u+P\right)^{*}=0,$$

and hence $\sum_{u\in U_A}\overline{\lambda_u}u^{-1}=0$ by semisimplicity. On the other hand, according to [3, Lemma 2.2], given $x\in A$ and $\varepsilon>0$, there exists a family $\{\lambda_u\}_{u\in U_A}$ of real (respectively, complex) numbers satisfying $\sum_{u\in U_A}|\lambda_u|<\|x\|+\varepsilon$ and $\sum_{u\in U_A}\lambda_u u=x$. It follows that

$$x=\sum_{u\in U_A}\lambda_u u\to x^{*}:=\sum_{u\in U_A}\overline{\lambda_u}u^{-1}$$

is a well-defined mapping from A to A, which actually becomes an isometric linear (respectively, conjugate-linear) algebra involution on A satisfying $u^*=u^{-1}$ for every $u\in U_A$.

(2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1). These implications follow from [1, Proposition 4.8]. $\square$

Acknowledgments

The authors are grateful to G. Dales, M. Neumann, and A.M. Peralta for their interesting remarks concerning the matter of the paper.

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